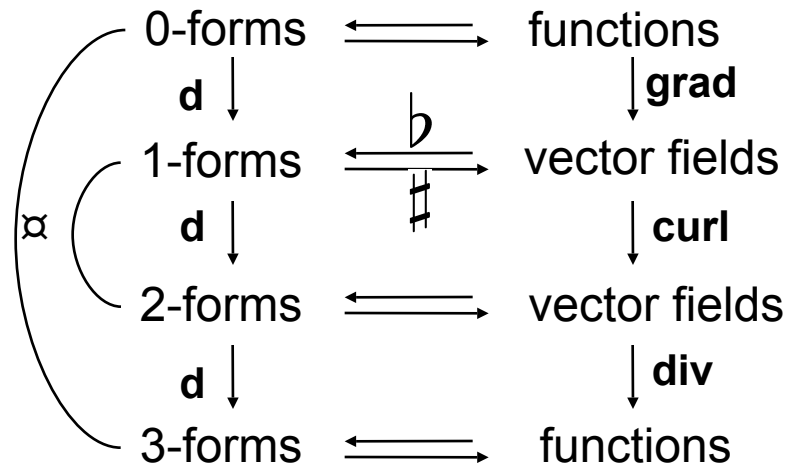


# Discrete Exterior Calculus

Notes from  
Anil Hirani's PhD Thesis

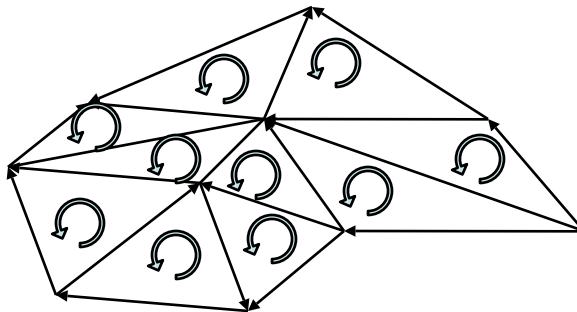
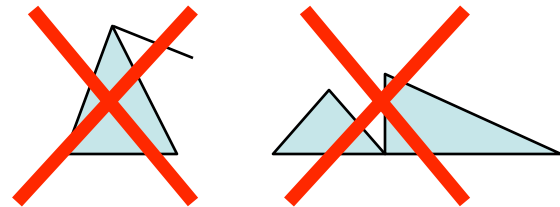
# Idea

- Discrete counterpart of smooth exterior calculus.
- Novelty: Simultaneous presence of forms and vector fields



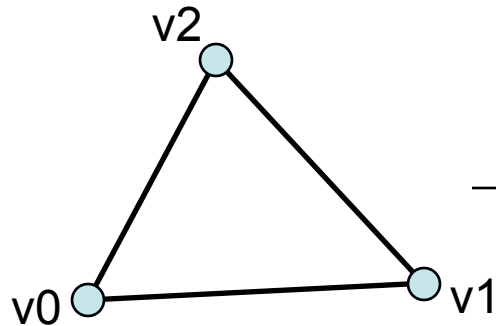
# Basics

- **Simplicial complex**
  - Collection of  $\sigma^0$  (vertices),  $\sigma^1$  (edges),  $\sigma^2$  (triangles)
- **Manifold-like**
  - Like a valid triangulation.
- **Oriented**

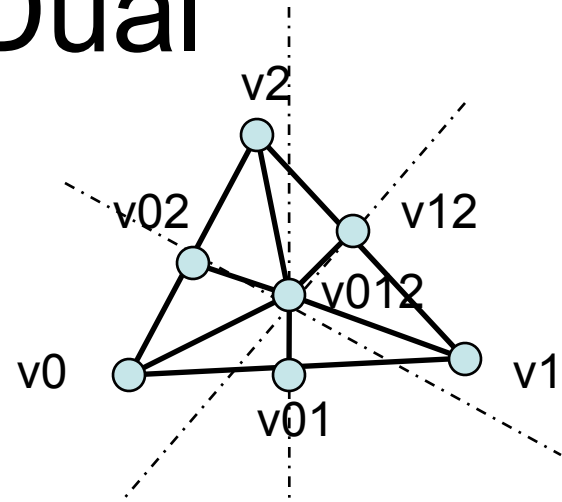


“Primal”

# Primal vs. Dual



circumcentric  
subdivision



$$\sigma^0 : v_0, v_1, v_2$$

$$\sigma^1 : [0, 1], [1, 2], [2, 0]$$

$$\sigma^2 : [0, 1, 2]$$

$$\sigma^0 : v_0, v_1, v_2, v_{01}, v_{12}, v_{02}, v_{012}$$

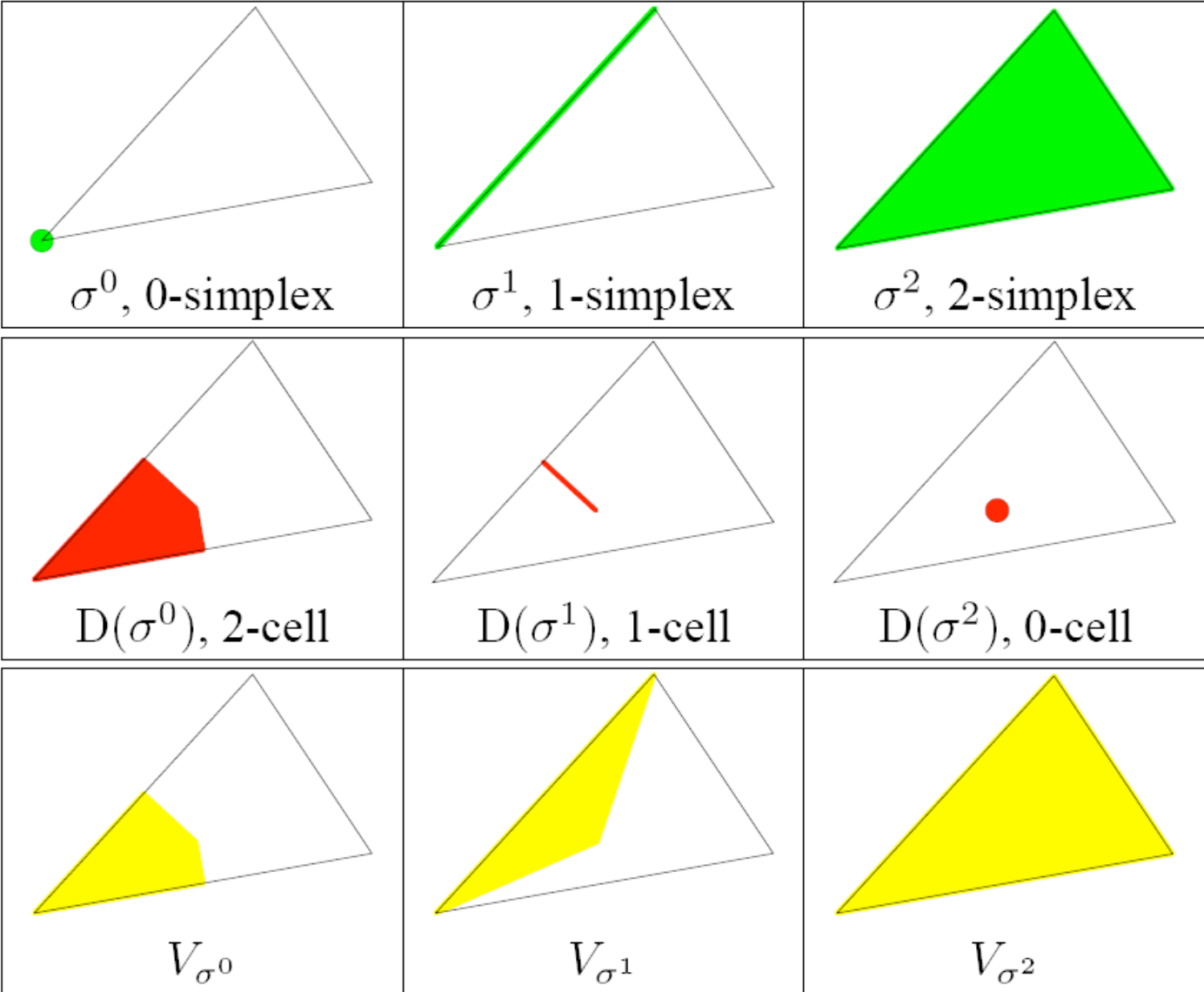
$$\sigma^1 : [0, 01], [01, 1], [1, 12], [12, 2], [2, 02], [02, 0], [0, 012], [01, 012], [1, 012], [12, 012], [2, 012], [02, 012]$$

$$\sigma^2 : [0, 01, 012], [01, 012, 1], [012, 1, 12], [012, 12, 2], [012, 2, 02], [0, 012, 02]$$

“well-centered”

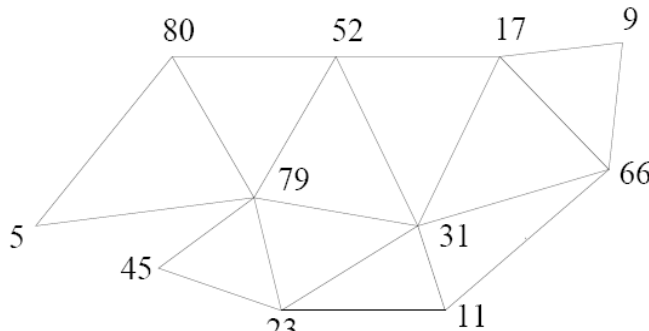
# Primal vs. Dual

**primal**  
 $\phi(\sigma, D(\sigma))$  &  
 $\phi(D(\sigma), \sigma):$   
 Interpolation  
 functions  
**dual**  
 •(p, n-p) relation



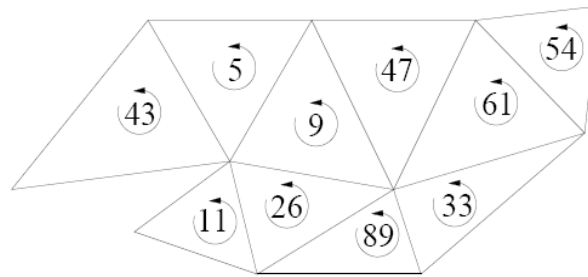
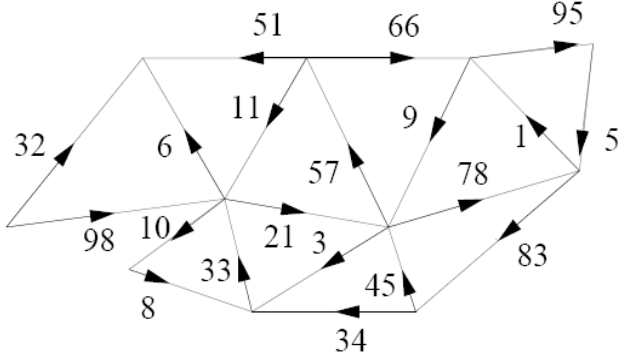
# Differential forms

- Smooth:  
“Integration of a p-form on a p-manifold”
- Discrete:  
“Evaluation of a discrete p-form on a p-chain”



Map from p-chains to  $\mathbb{R}$  (cochains)

Like an array of p-simplices of a complex  $K$



$$\Omega_d^p(K)$$

$$\Omega_d^p(\star K)$$

# Operators: Boundary Operator

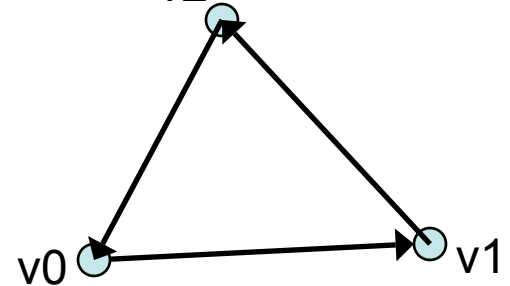
Boundary operator on a simplex

$$\sigma^p = [v_0, v_1, \dots, v_p]$$

$$\partial_p \sigma^p = \sum_{i=0}^p (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_p]$$

e.g.  $p = 2$ , a triangle

$$\partial_2 \sigma^2 = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

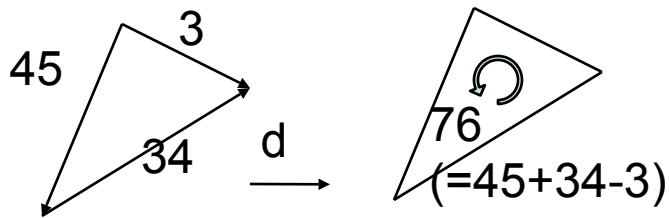


# Operators: Disc. Ext. Derivative

- Coboundary operator = Disc. Ext. Derivative

$$\langle \delta^p \alpha^p, c_{p+1} \rangle = \langle \alpha^p, \partial_{p+1} c_{p+1} \rangle$$

- Disc. Ext. derivative  $d = \delta: \Omega_d^p(K) \rightarrow \Omega_d^{p+1}(K)$



“map from a p-form (evaluated on the edge) to a (p+1)-form (evaluated on the triangle)”

- Stokes' Theorem:  $\langle d\alpha, c \rangle = \langle \alpha, \partial c \rangle$

# Operators: Hodge Star $\star$

- Smooth case:  $\star$  is an isomorphism between the space of  $p$ -forms and  $(n-p)$ -forms
- Discrete case:

**Reminder:** dual of a  $p$ -simplex is an  $(n-p)$ -cell

$$\star : \Omega_d^p(K) \rightarrow \Omega_d^{(n-p)}(\star K)$$

For a  $p$ -simplex  $\sigma^p$  and a discrete  $p$ -form  $\alpha^p$ :

$$\frac{1}{|\star\sigma^p|} \langle \star\alpha, \star\sigma^p \rangle := \frac{1}{\sigma^p} \langle \alpha, \sigma^p \rangle$$

# Operators: Codifferential

- “inverse” of discrete exterior derivative

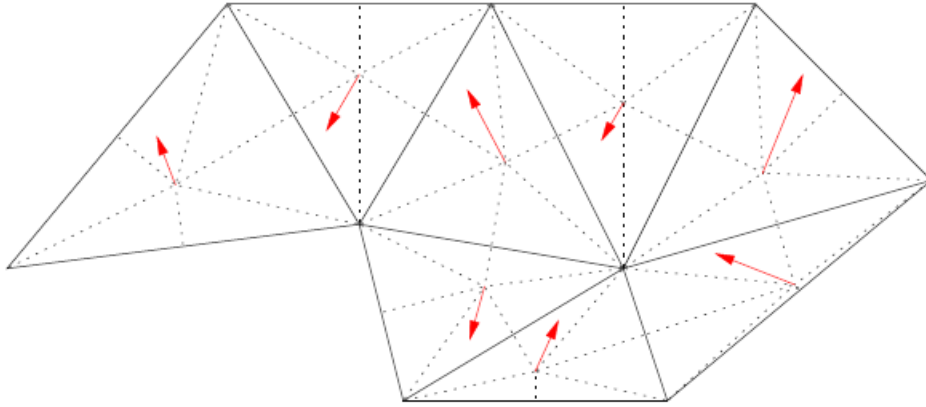
$$\delta : \Omega_d^{p+1}(K) \rightarrow \Omega_d^p(K)$$

- $\beta$  :  $(p+1)$ -discrete form

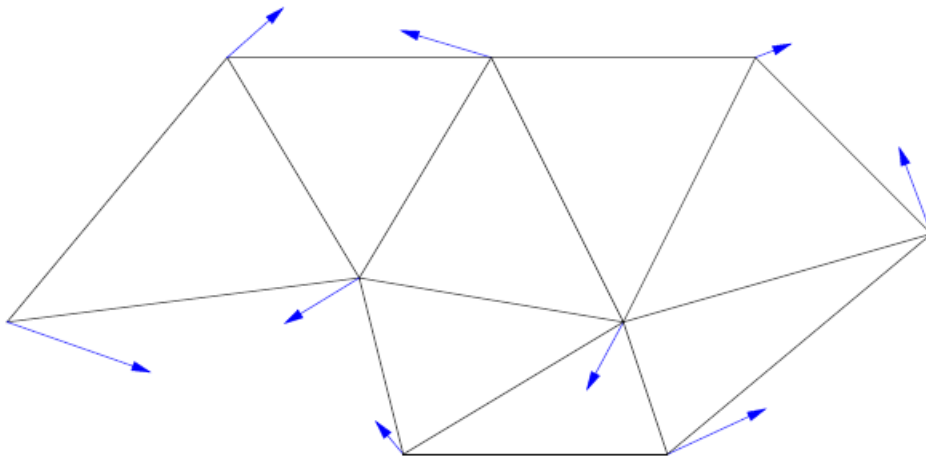
$$\delta\beta = (-1)^{np+1} * d * \beta$$

- For a 0-form:  $\delta(\Omega_d^0(K)) = 0$

# Discrete Vector Fields



Can interpolate to:  
 dual  $\begin{cases} D : \sum X \phi_{(D(\sigma), D(\sigma))} \\ \text{Or} \\ P : \sum X \phi_{(D(\sigma), \sigma)} \end{cases}$



primal  $\begin{cases} D : \sum X \phi_{(\sigma, D(\sigma))} \\ \text{Or} \\ P : \sum X \phi_{(\sigma, \sigma)} \end{cases}$

+

---

4 types of vector fields!

# Operators: Sharp<sup>#</sup> & Flat<sup>b</sup>

Smooth case:

Given a manifold with a metric  $\langle\langle, \rangle\rangle$  (Riemannian),  
X: vector field,  $v$ : tangent vector,  $\alpha$ : 1-form

**Sharp** : 1 forms  $\rightarrow$  vector fields

$$\langle\langle \alpha^{\#}, v \rangle\rangle = \alpha(v)$$

**Flat** : vector fields  $\rightarrow$  1 forms

$$\langle\langle X, v \rangle\rangle = X^b(v)$$

Sharp and Flat are inverses!

# Operators: Sharp & Flat

- Discrete flats:

- 2 types of vector fields
- 2 interpolation methods

× • 2 types of forms

8 different flat operators :

$b_{ppp}, b_{pdp}, b_{ppd}, b_{pdd},$   
 $b_{dpp}, b_{ddp}, b_{dpd}, b_{ddd}$

- Discrete sharps:

– Ad-hoc definitions with no interpolation

2 vecfields x 2 forms = 4 sharp operators

- Inverse problem!

$\#_{pp}, \#_{pd}, \#_{dp}, \#_{dd}$

# Operators: Discrete Flat

- $Y$  smooth vector field,  $r$  a smooth curve :

$$\int_r Y^b = \int_0^L Y(r(s)) \cdot \dot{r}(s) ds$$

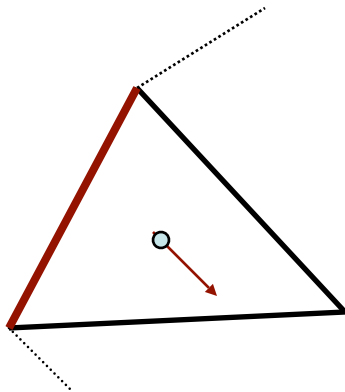
- $X$ : discrete vector field,  $r$ : edge (pw-sm. curve)

$$\langle X^\sharp, r \rangle := \int_r (\sum X \phi)^b$$

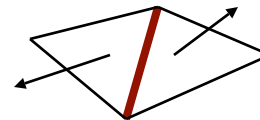
$$= \underbrace{\sum X \phi}_{\text{Vec field at an edge}} \cdot \underbrace{\vec{r}}_{\text{Vec along } r \text{ with length}=|r|}$$

Vec field at an edge    Vec along  $r$  with length= $|r|$

$b$  dpp



Problem:



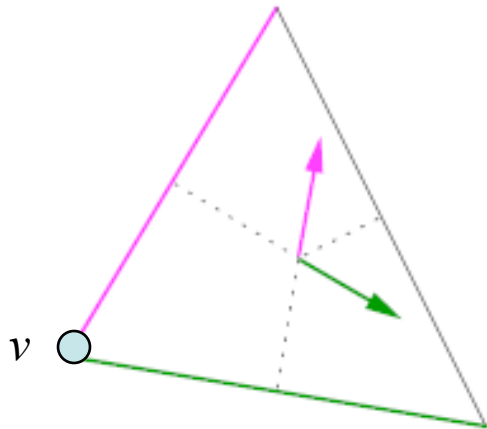
Solution: weighted averages (sup. vol.)

# Operations: Discrete Sharp

$\#_{pd}$

- Given  $f$ : 0-form,  $v$ : any vertex of a triangle

$$\langle (df)^\#, \star\sigma^2 \rangle := \sum_{\sigma^0 \prec \sigma^2} (f(\sigma^0) - f(v)) \nabla \phi_{\sigma, \sigma}$$



# Divergence

$$\operatorname{div}(X) = *d * X^b$$

Smooth divergence thm:

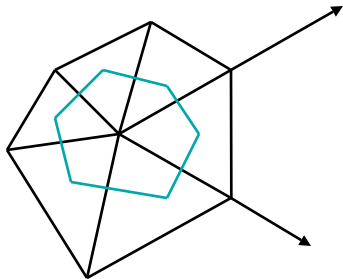
Integral of  $\operatorname{div}(X)$  over a region

Flux of  $X$  through boundary

$$\iiint \operatorname{div}(X) dV = \iint X \cdot n dS$$

Discrete div thm on dual cell

$$|\star\sigma^0| \langle \operatorname{div}(X), \sigma^0 \rangle = \sum_{\sigma^1 \succ \sigma^0} \sum_{\sigma^n \succ \sigma^1} |\star\sigma^1 \cap \sigma^n| \left( X(\star\sigma^n) \cdot \frac{\vec{\sigma}^1}{|\sigma^1|} \right)$$



# Divergence

$$\operatorname{div}(X) = *d * X^\flat$$

Smooth divergence thm:

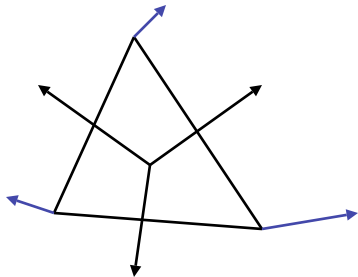
Integral of  $\operatorname{div}(X)$  over a region

Flux of  $X$  through boundary

$$\iiint \operatorname{div}(X) dV = \iint X \cdot n dS$$

Discrete div thm on a primal 2-simplex

$$|\sigma^n| \langle \operatorname{div}(X), \star\sigma^n \rangle = \sum_{\sigma^{n-1} \prec \sigma^n} |\sigma^{n-1}| \left( \sum_{\sigma^0 \prec \sigma^{n-1}} X(\sigma^0) \right) \cdot \frac{\star\sigma^{n-1}}{|\star\sigma^{n-1}|}$$



# Gradient

- Smooth:  $\nabla f = (df)^\#$
- Discrete PD-gradient:  
 $f$ : primal 0-form,  $v$  any vertex

$$\langle (df)^\#, \star \sigma^2 \rangle := \sum_{\sigma^0 \prec \sigma^2} (f(\sigma^0) - f(v)) \nabla \phi_{\sigma, \sigma}$$

# Curl

- 2D:  $*dX^b$

$$\langle \text{curl}(X), \sigma^0 \rangle = \frac{1}{|\star\sigma^0|} \sum_{\sigma^2 \succ \sigma^0} X(\star\sigma^2) \cdot \sigma_{outer}^1$$

- Polthier/Preuss '02

$$\langle \text{curl}(X), \sigma^0 \rangle = \frac{1}{2} \sum_{\sigma^2 \succ \sigma^0} X(\star\sigma^2) \cdot \sigma_{outer}^1$$

- 3D:  $(*dX^b)^\#$

# Laplace-Beltrami

- Smooth :  $\nabla^2 = \text{div} \cdot \text{curl} = \delta d$   
 (special case of Laplace-DeRham :  $\Delta = d\delta + \delta d$ )

- Discrete:  
 $\langle \Delta f, \sigma^0 \rangle = \langle (d\delta + \delta d)(f), \sigma^0 \rangle$   
 $= \langle \cancel{d\delta f} + \delta df, \sigma^0 \rangle$   
 $= - \langle *d * df, \sigma^0 \rangle$

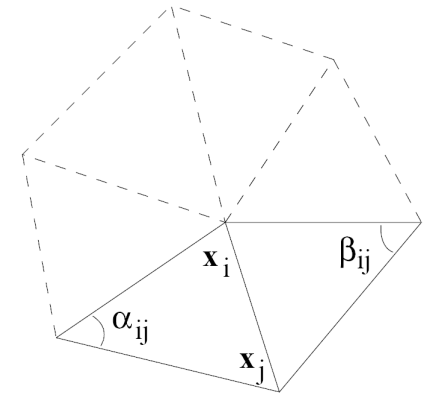
# Laplace-Beltrami

$$\langle \Delta f, \sigma^0 \rangle = - \langle *d * df, \sigma^0 \rangle$$

...

$$\langle \Delta f, \sigma^0 \rangle = \frac{1}{|\star \sigma^0|} \sum_{\sigma^1 = [\sigma^0, v]} \frac{|\star \sigma^1|}{|\sigma^1|} (f(v) - f(\sigma^0))$$

Meyer et al.02:



$$\Delta f(\mathbf{x}_i) = \frac{1}{2\mathcal{A}} \sum_{j \in N_1(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) (f(\mathbf{x}_i) - f(\mathbf{x}_j))$$